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Differential operators having Sobolev-type Laguerre polynomials as eigenfunctions: new developments

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Abstract

In this paper we consider polynomials, orthogonal with respect to an inner product which consists of the classical Laguerre inner product combined with two linear perturbations of Sobolev type at $x = 0$. We derive linear differential operators, of a specific form and usually of infinite order, having these polynomials as eigenfunctions. In the case, α is a nonnegative integer one of the operators is of finite order. © 2001 Elsevier Science B.V. All rights reserved.

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1. Introduction

In this paper we consider the Sobolev-type Laguerre polynomials $\{L_n^{\alpha,M_1,M_2}(x,l_1,l_2)\}_{n=0}^{\infty}$, orthogonal with respect to the inner product

$$\langle p,q\rangle = \frac{1}{\Gamma(\alpha+1)}\int_0^{\infty} p(x)q(x)x^{\alpha}\mathrm{e}^{-x}\,\mathrm{d}x + M_1 p^{(l_1)}(0)q^{(l_1)}(0) + M_2 p^{(l_2)}(0)q^{(l_2)}(0), \tag{1}$$

where $\alpha > -1$, $M_1 \geqslant 0$, $M_2 \geqslant 0$ and l_1 and l_2 are different nonnegative integers. Without loss of the generality we assume $l_1 < l_2$. These polynomials are generalizations of the Laguerre polynomials $\{L_n^{(\alpha)}(x)\}_{n=0}^{\infty}$, which are known to be eigenfunctions of the second-order linear differential operator

$$\boldsymbol{L}^{(\alpha)} = -x\boldsymbol{D}^2 - (\alpha+1-x)\boldsymbol{D}$$

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($\boldsymbol{D} = \mathrm{d}/\mathrm{d}x$) with eigenvalues $\lambda_n = n$. The purpose of this paper is to find linear differential operators, possibly of infinite order and of the form

$$\boldsymbol{L}^{(\alpha)} + M_1\boldsymbol{A}^{(\alpha,l_1)} + M_2\boldsymbol{A}^{(\alpha,l_2)} + M_1M_2\boldsymbol{C}^{(\alpha,l_1,l_2)} \tag{2}$$

with eigenvalues of the form

$$\{n + M_1\alpha_n^{(\alpha,l_1)} + M_2\alpha_n^{(\alpha,l_2)} + M_1M_2\gamma_n^{(\alpha,l_1,l_2)}\}_{n=0}^{\infty}, \tag{3}$$

such that the polynomials $\{L_n^{\alpha,M_1,M_2}(x,l_1,l_2)\}_{n=0}^{\infty}$ are eigenfunctions of (2) with eigenvalues (3). Especially, we are interested in such linear differential operators, when they are of finite order. In a number of special cases, this problem has been considered before (see [3] for a complete survey). Here we only mention the case $M_2 = 0$, which for $l_1 = 0$ has been treated in [8] (see also [1,6]) and, in general, in [2], and the case $l_1 = 0$, $l_2 = 1$, which was studied in [9]. In all these cases, a finite-order operator exists if and only if α is a nonnegative integer ($\alpha \in \mathbb{N}$).

As an application in [5, Section 5.1] (see also [7]), it was shown that there exist linear differential operators $\boldsymbol{A}^{(\alpha,l_1)}$, $\boldsymbol{A}^{(\alpha,l_2)}$, $\boldsymbol{C}^{(\alpha,l_1,l_2)}$ (usually of infinite order) and numbers

$$\{\alpha_n^{(\alpha,l_1)}\}_{n=0}^{\infty}, \quad \{\alpha_n^{(\alpha,l_2)}\}_{n=0}^{\infty}, \quad \{\gamma_n^{(\alpha,l_1,l_2)}\}_{n=0}^{\infty}$$

such that the polynomials $\{L_n^{\alpha,M_1,M_2}(x,l_1,l_2)\}_{n=0}^{\infty}$ are solutions of the differential equation

$$\begin{aligned}&[(\boldsymbol{L}^{(\alpha)} - n\boldsymbol{I}) + M_1(\boldsymbol{A}^{(\alpha,l_1)} - \alpha_n^{(\alpha,l_1)}\boldsymbol{I}) + M_2(\boldsymbol{A}^{(\alpha,l_2)} - \alpha_n^{(\alpha,l_2)}\boldsymbol{I})\\ &\quad + M_1M_2(\boldsymbol{C}^{(\alpha,l_1,l_2)} - \gamma_n^{(\alpha,l_1,l_2)}\boldsymbol{I})]y(x) = 0.\end{aligned} \tag{4}$$

Here

$$\boldsymbol{A}^{(\alpha,l)} = \sum_{i=1}^{\infty} a_i(x;\alpha,l)\boldsymbol{D}^i, \quad l \in \{l_1,l_2\}, \quad \boldsymbol{C}^{(\alpha,l_1,l_2)} = \sum_{i=1}^{\infty} c_i(x;\alpha,l_1,l_2)\boldsymbol{D}^i.$$

Further we have to take $\alpha_0^{(\alpha,l_1)} = \alpha_0^{(\alpha,l_2)} = \gamma_0^{(\alpha,l_1,l_2)} = 0$ and the values

$$\{\alpha_n^{(\alpha,l_1)}\}_{n=1}^{l_1} \quad (\text{if } l_1 > 0), \quad \{\alpha_n^{(\alpha,l_2)}\}_{n=1}^{l_2}, \quad \{\gamma_n^{(\alpha,l_1,l_2)}\}_{n=1}^{l_1} \quad (\text{if } l_1 > 0)$$

can be chosen arbitrarily; for the other values formulas are given. To each choice of the arbitrary values corresponds precisely one linear differential operator of form (2), usually of infinite order. In this paper we will show that if all the arbitrary values are chosen to be 0 and further $\alpha \in \mathbb{N}$, then the corresponding operators $\boldsymbol{A}_0^{(\alpha,l_1)}, \boldsymbol{A}_0^{(\alpha,l_2)}$ and $\boldsymbol{C}_0^{(\alpha,l_1,l_2)}$ are of finite order: $\boldsymbol{A}_0^{(\alpha,l)}$ is of order $2\alpha + 4l + 4$, $l \in \{l_1,l_2\}$ and $\boldsymbol{C}_0^{(\alpha,l_1,l_2)}$ is of order $4\alpha + 4l_1 + 4l_2 + 6$. This proves a conjecture in [5, Section 5.1]. Further we will show that any other choice of the arbitrary values will lead to an operator of infinite order and also that if $\alpha \notin \mathbb{N}$, then the operator is of infinite order for any choice of the arbitrary values.

2. Representation of the polynomials

In [5, Section 2] a representation for this type of orthogonal polynomials is given, which becomes in this case

$$L_n^{\alpha,M_1,M_2}(x,l_1,l_2) = L_n^{(\alpha)}(x) + M_1Q_n^{\alpha}(x,l_1) + M_2Q_n^{\alpha}(x,l_2) + M_1M_2S_n^{\alpha}(x,l_1,l_2). \tag{5}$$

Here for $l \in \{l_1, l_2\}$

$$Q_n^\alpha(x,l) = K_{n-1}^{(l,l)}(0,0)L_n^{(\alpha)}(x) - (-1)^l L_{n-l}^{(\alpha+l)}(0)K_{n-1}^{(0,l)}(x,0)$$

and

$$S_n^\alpha(x,l_1,l_2) = \begin{vmatrix} L_n^{(\alpha)}(x) & K_{n-1}^{(0,l_1)}(x,0) & K_{n-1}^{(0,l_2)}(x,0) \\ (-1)^{l_1}\begin{pmatrix} n+\alpha \\ n-l_1 \end{pmatrix} & K_{n-1}^{(l_1,l_1)}(0,0) & K_{n-1}^{(l_1,l_2)}(0,0) \\ (-1)^{l_2}\begin{pmatrix} n+\alpha \\ n-l_2 \end{pmatrix} & K_{n-1}^{(l_1,l_2)}(0,0) & K_{n-1}^{(l_2,l_2)}(0,0) \end{vmatrix},$$

where

$$K_n^{(r,s)}(x,y) = \sum_{k=0}^{n} \frac{\boldsymbol{D}^r L_k^{(\alpha)}(x)\boldsymbol{D}^s L_k^{(\alpha)}(y)}{\begin{pmatrix} k+\alpha \\ k \end{pmatrix}}, \quad r,s,n \in \mathbb{N}. \tag{6}$$

Note that $K_n^{(r,s)}(x,y) = 0$ if $n < \max\{r,s\}$. In order to obtain a useful representation of $K_{n-1}^{(0,l)}(x,0)$ we prove the following:

Lemma 1.

$$L_{n-k}^{(\alpha)}(x) = \sum_{j=0}^{n-k} \begin{pmatrix} -k \\ j \end{pmatrix} L_{n-k-j}^{(\alpha+k+j)}(x), \quad k,n \in \mathbb{N}, \; k \leqslant n.$$

Proof. From the generating function

$$\sum_{n=0}^{\infty} L_n^{(\alpha)}(x)t^n = (1-t)^{-\alpha-1}\exp\left(\frac{-xt}{1-t}\right),$$

it easily follows that for arbitrary p

$$L_n^{(\alpha-p)}(x) = \sum_{j=0}^{n} \begin{pmatrix} p \\ j \end{pmatrix} (-1)^j L_{n-j}^{(\alpha)}(x),$$

hence for $k,n \in \mathbb{N}, \; k \leqslant n$

$$L_{n-k}^{(\alpha+k)}(x) = \sum_{j=0}^{n-k} \begin{pmatrix} -k \\ j \end{pmatrix} (-1)^j L_{n-k-j}^{(\alpha)}(x).$$

This system of equations can be inverted to obtain the desired result. $\square$

By using Lemma 1, the following formula can be derived from (6) by straightforward calculation:

$$K_{n-1}^{(0,l)}(x,0) = \frac{(-1)^l l!}{(\alpha+1)_l} \sum_{k=1}^{l+1} (-1)^{k-1} \begin{pmatrix} n-k \\ l-k+1 \end{pmatrix} L_{n-k}^{(\alpha+k)}(x), \quad n \geqslant l+1$$

and, hence,

$$K_{n-1}^{(m,l)}(0,0) = \frac{(-1)^{l+m} l!}{(\alpha+1)_l} \sum_{k=1}^{l+1} (-1)^{k-1} \binom{n-k}{l-k+1} \binom{n+\alpha}{n-m-k}. \tag{7}$$

We derive another representation of $K_{n-1}^{(m,l)}(0,0)$. We use (6) and the formula

$$\sum_{k=0}^{n} \frac{(a)_k (b)_k}{(c)_k k!} = \frac{(a+1)_n}{n!} \, {}_3F_2 \left(\left. \begin{matrix} -n,\ a,\ c-b \\ c,\ a+1 \end{matrix} \right| 1 \right),$$

(see [9, Lemma 4]) to obtain ($l \leqslant m$)

$$\begin{aligned} K_{n-1}^{(m,l)}(0,0) &= \frac{(-1)^{l+m} m!}{(\alpha+1)_l (m-l)!} \sum_{v=0}^{n-m-1} \frac{(m+1)_v (\alpha+m+1)_v}{(m-l+1)_v v!} \\ &= \frac{(-1)^{l+m} n!}{(\alpha+1)_l (m-l)! (n-m-1)! (m+1)} \, {}_3F_2 \left(\left. \begin{matrix} m+1-n, & m+1, & -l-\alpha \\ & m-l+1, & m+2 \end{matrix} \right| 1 \right) \\ &= \frac{(-1)^{l+m}}{(\alpha+1)_l} \sum_{v=0}^{\infty} \frac{(-1)^v (-l-\alpha)_v \prod_{j=0}^{m+v} (n-j)}{v! (m-l+v)! (m+v+1)}, \end{aligned} \tag{8}$$

which can be used for all $n \in \mathbb{N}$. We will need the following lemma.

Lemma 2. *Let $p, q, n \in \mathbb{Z}$ with $p \leqslant q \leqslant n-1$. Then*

$$\sum_{i=q+1}^{n} \prod_{j=p+1}^{q} (i-j) = \frac{\prod_{j=p}^{q} (n-j)}{q-p+1} \tag{9}$$

and if $p \leqslant s \leqslant q$

$$\sum_{i=q+1}^{n} \left[\prod_{j=p+1}^{q} (i-j) \prod_{j=p+1}^{s} (i-j) \right] = \frac{\prod_{j=p}^{q} (n-j)}{q+s-2p+1} \tilde{\pi}(n), \tag{10}$$

where $\tilde{\pi}(x)$ is a monic polynomial of degree $s-p$.

Proof. Formula (9) is a direct consequence of the identity

$$\prod_{j=p}^{q} (i-j) - \prod_{j=p}^{q} (i-1-j) = (q-p+1) \prod_{j=p+1}^{q} (i-j).$$

Putting

$$\Sigma(n) := \sum_{i=q+1}^{n} \left[\prod_{j=p+1}^{q} (i-j) \prod_{j=p+1}^{s} (i-j) \right],$$

we note that $\Sigma(n)$ is a polynomial $\Sigma(x)$ in $x=n$ of degree $q+s-2p+1$ with leading coefficient $1/(q+s-2p+1)$. Moreover, since

$$\Sigma(x)-\Sigma(x-1)=\left[\prod_{j=p+1}^{q}(x-j)\prod_{j=p+1}^{s}(x-j)\right],$$

we may conclude that $\Sigma(x)=\Sigma(x-1)$ for $x=p+1,p+2,\ldots,q$ and thus $\Sigma(x)=0$, if $x=p,p+1,\ldots,q$. □

3. The eigenvalues

From [5], it follows that we have to take $\alpha_0^{(\alpha,l_1)}=\alpha_0^{(\alpha,l_2)}=\gamma_0^{(\alpha,l_1,l_2)}=0$ and the values

$$\{\alpha_n^{(\alpha,l_1)}\}_{n=1}^{l_1}\quad(\text{if } l_1>0),\quad \{\alpha_n^{(\alpha,l_2)}\}_{n=1}^{l_2},\quad \{\gamma_n^{(\alpha,l_1,l_2)}\}_{n=1}^{l_1}\quad(\text{if } l_1>0)$$

can be chosen arbitrarily, whereas for $l\in\{l_1,l_2\}$ and $n>l$

$$\alpha_n^{(\alpha,l)}=\alpha_l^{(\alpha,l)}+\sum_{j=l+1}^{n}K_{j-1}^{(l,l)}(0,0).$$

Further for $n\in\{l_1+1,l_1+2,\ldots,l_2\}$

$$\gamma_n^{(\alpha,l_1,l_2)}=\gamma_{l_1}^{(\alpha,l_1,l_2)}+\sum_{j=l_1+1}^{n}(\alpha_j^{(\alpha,l_2)}-\alpha_{j-1}^{(\alpha,l_2)})K_{j-1}^{(l_1,l_1)}(0,0)$$

and for $n>l_2$

$$\gamma_n^{(\alpha,l_1,l_2)}=\gamma_{l_2}^{(\alpha,l_1,l_2)}+\sum_{j=l_2+1}^{n}\begin{vmatrix}K_{j-1}^{(l_1,l_1)}(0,0) & K_{j-1}^{(l_1,l_2)}(0,0)\\ K_{j-1}^{(l_1,l_2)}(0,0) & K_{j-1}^{(l_2,l_2)}(0,0)\end{vmatrix}.$$

If we choose all the arbitrary values to be 0, then for $l\in\{l_1,l_2\}$

$$\alpha_n^{(\alpha,l)}=\sum_{j=1}^{n}K_{j-1}^{(l,l)}(0,0),\quad n\in\mathbb{N} \tag{11}$$

and

$$\gamma_n^{(\alpha,l_1,l_2)}=\sum_{j=l_2+1}^{n}\begin{vmatrix}K_{j-1}^{(l_1,l_1)}(0,0) & K_{j-1}^{(l_1,l_2)}(0,0)\\ K_{j-1}^{(l_1,l_2)}(0,0) & K_{j-1}^{(l_2,l_2)}(0,0)\end{vmatrix},\quad n\in\mathbb{N}. \tag{12}$$

4. Some functions and polynomials in $x=n$

- It follows from (8) that $K_{n-1}^{(m,l)}(0,0)$ is a function $K^{(m,l)}(x)$ in $x=n$ with

$$K^{(m,l)}(x)=\frac{(-1)^{l+m}}{(\alpha+1)_l}\sum_{v=0}^{\infty}\frac{(-1)^v(-l-\alpha)_v\prod_{j=0}^{m+v}(x-j)}{v!(m-l+v)!(m+v+1)},\quad l\leqslant m.$$

If $\alpha\notin\mathbb{N}$, then this representation is valid for $\mathrm{Re}\,x>-\alpha-1$.

If $\alpha \in \mathbb{N}$, then $K^{(m,l)}(x)$ is a polynomial of degree $\alpha+m+l+1$ and

$$\lim_{x\to\infty} \frac{K^{(m,l)}(x)}{x^{\alpha+m+l+1}} = \frac{(-1)^{l+m}\alpha!}{(\alpha+l+m+1)(\alpha+l)!(\alpha+m)!}.$$

Further by (7) we have $K^{(m,l)}(x)=0$, if $x=-\alpha,-\alpha+1,\ldots,\max\{l,m\}$.

- Also, from (8), (11) and (9), we find that $\alpha_n^{(\alpha,l)}$ is a function $\alpha^{(\alpha,l)}(x)$ in $x=n$ with

$$\alpha^{(\alpha,l)}(x) = \frac{1}{(\alpha+1)_l}\sum_{v=0}^{\infty} \frac{(-1)^v(-l-\alpha)_v \prod_{j=-1}^{l+v}(x-j)}{v!v!(l+v+1)(l+v+2)}. \tag{13}$$

If $\alpha \notin \mathbb{N}$, then this representation is valid for $\mathrm{Re}\, x > -\alpha-2$.
If $\alpha \in \mathbb{N}$, then $\alpha^{(\alpha,l)}(x)$ is a polynomial of degree $\alpha+2l+2$ and

$$\lim_{x\to\infty} \frac{\alpha^{(\alpha,l)}(x)}{x^{\alpha+2l+2}} = \frac{\alpha!}{(\alpha+2l+1)(\alpha+2l+2)[(\alpha+l)!]^2}. \tag{14}$$

Since $\alpha^{(\alpha,l)}(0)=0$ and

$$\alpha^{(\alpha,l)}(x)-\alpha^{(\alpha,l)}(x-1)=K^{(l,l)}(x)$$

by (7), we may conclude that $\alpha^{(\alpha,l)}(x)=\alpha^{(\alpha,l)}(x-1)$ for $x=-\alpha,-\alpha+1,\ldots,l$ and thus $\alpha^{(\alpha,l)}(x)=0$, if $x=-\alpha-1,-\alpha,\ldots,l$.

- From (8), (12) and (10), we find that $\gamma_n^{(\alpha,l_1,l_2)}$ is a function $\gamma^{(\alpha,l_1,l_2)}(x)$ in $x=n$. If $\alpha\in\mathbb{N}$, then $\gamma^{(\alpha,l_1,l_2)}(x)$ is a polynomial of degree $2\alpha+2l_1+2l_2+3$ and

$$\lim_{x\to\infty} \frac{\gamma^{(\alpha,l_1,l_2)}(x)}{x^{2\alpha+2l_1+2l_2+3}} = \frac{(\alpha!)^2(l_2-l_1)^2}{(\alpha+2l_1+1)(\alpha+2l_2+1)(2\alpha+2l_1+2l_2+3)[(\alpha+l_1+l_2+1)(\alpha+l_1)!(\alpha+l_2)!]^2}. \tag{15}$$

Since $\gamma^{(\alpha,l_1,l_2)}(0)=0$ and

$$\gamma^{(\alpha,l_1,l_2)}(x)-\gamma^{(\alpha,l_1,l_2)}(x-1)=K^{(l_1,l_1)}(x)K^{(l_2,l_2)}(x)-(K^{(l_1,l_2)}(x))^2,$$

we may conclude that $\gamma^{(\alpha,l_1,l_2)}(x)=\gamma^{(\alpha,l_1,l_2)}(x-1)$ for $x=-\alpha,-\alpha+1,\ldots,l_2$ and thus $\gamma^{(\alpha,l_1,l_2)}(x)=0$, if $x=-\alpha-1,-\alpha,\ldots,l_2$.

- In the following, we will need two other functions in $x=n$.
First $\binom{n+\alpha}{n-l}$ is a function in $x=n$, which we denote by $\sigma^{l,\alpha}(x)$. We have

$$\sigma^{l,\alpha}(x) = \sum_{v=0}^{\infty} \frac{(-1)^v(-l-\alpha)_v \prod_{j=l}^{l+v-1}(x-j)}{v!v!}. \tag{16}$$

If $\alpha \notin \mathbb{N}$, then this representation is valid for $\mathrm{Re}\, x > -\alpha-1$.
If $\alpha\in\mathbb{N}$, then $\sigma^{l,\alpha}(x)$ is a polynomial of degree $\alpha+l$.
Also for each $k\in\{1,2,\ldots,l+1\}$$\binom{n-k}{l-k+1}$ is a polynomial in $x=n$ of degree $l+1-k$, for which we put $\pi^{k,l}(x)$ and

$$\pi^{k,l}(x) = \frac{\prod_{j=k}^{l}(x-j)}{(l-k+1)!}. \tag{17}$$

5. The operators

If we insert (5) into (4) with $y(x) = L_n^{\alpha,M_1,M_2}(x,l_1,l_2)$ and equate the coefficients of the corresponding powers of M_1 and M_2, then we obtain eight systems of equations from which we need

$$(\boldsymbol{A}^{(\alpha,l)} - \alpha_n^{(\alpha,l)}\boldsymbol{I})L_n^{(\alpha)}(x) + (\boldsymbol{L}^{(\alpha)} - n\boldsymbol{I})Q_n^{\alpha}(x,l) = 0, \quad l \in \{l_1, l_2\} \tag{18}$$

and

$$\begin{aligned}(\boldsymbol{C}^{(\alpha,l_1,l_2)} - \gamma_n^{(\alpha,l_1,l_2)}\boldsymbol{I})L_n^{(\alpha)}(x) + (\boldsymbol{L}^{(\alpha)} - n\boldsymbol{I})S_n^{\alpha}(x,l_1,l_2)\\ +(\boldsymbol{A}^{(\alpha,l_1)} - \alpha_n^{(\alpha,l_1)}\boldsymbol{I})Q_n^{\alpha}(x,l_2) + (\boldsymbol{A}^{(\alpha,l_2)} - \alpha_n^{(\alpha,l_2)}\boldsymbol{I})Q_n^{\alpha}(x,l_1) = 0,\end{aligned} \tag{19}$$

$n \in \mathbb{N}$. It is easy to derive that (see [2]) that

$$(\boldsymbol{L}^{(\alpha)} - n\boldsymbol{I})L_{n-k}^{(\alpha+k)}(x) = -k(L_{n-k}^{(\alpha+k)}(x) + L_{n-k-1}^{(\alpha+k+1)}(x));$$

hence,

$$(\boldsymbol{L}^{(\alpha)} - n\boldsymbol{I})K_{n-1}^{(0,l)}(x,0) = \frac{(-1)^l l!}{(\alpha+1)_l}\sum_{k=1}^{l+1}(-1)^k\binom{n-k}{l-k+1}k(L_{n-k}^{(\alpha+k)}(x) + L_{n-k-1}^{(\alpha+k+1)}(x)), \tag{20}$$

which by (18) leads to

$$\begin{aligned}&(\boldsymbol{A}^{(\alpha,l)} - \alpha_n^{(\alpha,l)}\boldsymbol{I})L_n^{(\alpha)}(x)\\ &\quad = \frac{\binom{n+\alpha}{n-l}l!}{(\alpha+1)_l}\sum_{k=1}^{l+1}(-1)^k k\binom{n-k}{l-k+1}(L_{n-k}^{(\alpha+k)}(x) + L_{n-k-1}^{(\alpha+k+1)}(x)).\end{aligned}$$

We now construct a linear differential operator, which has the same working on all the Laguerre polynomials as $\boldsymbol{A}^{(\alpha,l)} = \boldsymbol{A}_0^{(\alpha,l)} = \sum_{i=1}^{\infty} a_{i,0}(x;\alpha,l)\boldsymbol{D}^i$, where $\boldsymbol{A}_0^{(\alpha,l)}$ denotes the operator which corresponds to the $\{\alpha_n^{(\alpha,l)}\}_{n=0}^{\infty}$ given by (11). Since such an operator is uniquely determined, we find

$$\boldsymbol{A}_0^{(\alpha,l)} = \alpha^{(\alpha,l)}(\boldsymbol{L}^{(\alpha)}) + \frac{l!}{(\alpha+1)_l}\sum_{k=1}^{l+1}k(\boldsymbol{D}^k - \boldsymbol{D}^{k+1})\pi^{k,l}(\boldsymbol{L}^{(\alpha)})\sigma^{l,\alpha}(\boldsymbol{L}^{(\alpha)}) \tag{21}$$

using (13), (16) and (17), where in the products for $x-j$, we substitute $\boldsymbol{L}^{(\alpha)} - j\boldsymbol{I}$.

If $\alpha \notin \mathbb{N}$, then the operators $\alpha^{(\alpha,l)}(\boldsymbol{L}^{(\alpha)})$ and $\sigma^{l,\alpha}(\boldsymbol{L}^{(\alpha)})$ are of infinite order. In [3, p. 115], it has already been shown, that the operator $\boldsymbol{A}_0^{(\alpha,l)}$ is of infinite order in this case.

If $\alpha \in \mathbb{N}$, then the order of $\alpha^{(\alpha,l)}(\boldsymbol{L}^{(\alpha)})$ is $2\alpha+4l+4$ and the order of $(\boldsymbol{D}^k - \boldsymbol{D}^{k+1})\pi^{k,l}(\boldsymbol{L}^{(\alpha)})\sigma^{l,\alpha}(\boldsymbol{L}^{(\alpha)})$ is $2\alpha+4l+3-k$, which shows that $\boldsymbol{A}_0^{(\alpha,l)}$ is of order $2\alpha+4l+4$ and by (14), (see [2]),

$$a_{2\alpha+4l+4,0}(x;\alpha,l) = \frac{\alpha!(-x)^{\alpha+2l+2}}{(\alpha+2l+1)(\alpha+2l+2)[(\alpha+l)!]^2}, \quad l \in \{l_1,l_2\}.$$

Further, we construct a linear differential operator, which has the same working on all the Laguerre polynomials as $\boldsymbol{C}^{(\alpha,l_1,l_2)} = \boldsymbol{C}_0^{(\alpha,l_1,l_2)} = \sum_{i=1}^{\infty} c_{i,0}(x;\alpha,l_1,l_2)\boldsymbol{D}^i$, where $\boldsymbol{C}_0^{(\alpha,l_1,l_2)}$ denotes the operator which corresponds to the $\{\gamma_n^{(\alpha,l_1,l_2)}\}_{n=0}^{\infty}$ given by (12). Therefore, we investigate the different terms in (19). First,

$$(\boldsymbol{L}^{(\alpha)} - n\boldsymbol{I})S_n^{\alpha}(x,l_1,l_2) = \rho_1(n)(\boldsymbol{L}^{(\alpha)} - n\boldsymbol{I})K_{n-1}^{(0,l_1)}(x,0) + \rho_2(n)(\boldsymbol{L}^{(\alpha)} - n\boldsymbol{I})K_{n-1}^{(0,l_2)}(x,0),$$

where $\rho_1(x) = (-1)^{l_2}\ K^{(l_1,l_2)}(x)\sigma^{l_2,\alpha}(x) - (-1)^{l_1}\ K^{(l_2,l_2)}(x)\sigma^{l_1,\alpha}(x)$ and similarly $\rho_2(x)$ are functions which in the case $\alpha \in \mathbb{N}$ are polynomials, respectively, of degree $2\alpha+2l_2+1+l_1$ and $2\alpha+2l_1+1+l_2$. It is easy to construct the unique linear differential operator $\boldsymbol{U}^{(\alpha,l_1,l_2)}$ such that

$$\boldsymbol{U}^{(\alpha,l_1,l_2)}L_n^{(\alpha)}(x) = -(\boldsymbol{L}^{(\alpha)} - n\boldsymbol{I})S_n^{\alpha}(x,l_1,l_2), \quad n \in \mathbb{N}.$$

In fact,

$$\boldsymbol{U}^{(\alpha,l_1,l_2)} = -\frac{(-1)^{l_1}l_1!}{(\alpha+1)_{l_1}}\sum_{k=1}^{l_1+1} k(\boldsymbol{D}^k - \boldsymbol{D}^{k+1})\pi^{k,l_1}(\boldsymbol{L}^{(\alpha)})\rho_1(\boldsymbol{L}^{(\alpha)})$$
$$-\frac{(-1)^{l_2}l_2!}{(\alpha+1)_{l_2}}\sum_{k=1}^{l_2+1} k(\boldsymbol{D}^k - \boldsymbol{D}^{k+1})\pi^{k,l_2}(\boldsymbol{L}^{(\alpha)})\rho_2(\boldsymbol{L}^{(\alpha)}).$$

If $\alpha \in \mathbb{N}$, then the operator $\boldsymbol{U}^{(\alpha,l_1,l_2)}$ is of order $4\alpha + 4l_1 + 4l_2 + 4$. We now construct the unique linear differential operator $\boldsymbol{V}^{(\alpha,l_1,l_2)}$ such that

$$\boldsymbol{V}^{(\alpha,l_1,l_2)}L_n^{(\alpha)}(x) = -(\boldsymbol{A}_0^{(\alpha,l_1)} - \alpha_n^{(\alpha,l_1)}\boldsymbol{I})Q_n^{\alpha}(x,l_2), \quad n \in \mathbb{N}.$$

Since,

$$Q_n^{\alpha}(x,l_2) = K_{n-1}^{(l_2,l_2)}(0,0)L_n^{(\alpha)}(x) - (-1)^{l_2}\binom{n+\alpha}{n-l_2}K_{n-1}^{(0,l_2)}(x,0),$$

we have

$$\boldsymbol{V}^{(\alpha,l_1,l_2)} = -\frac{l_1!}{(\alpha+1)_{l_1}}\sum_{k=1}^{l_1+1} k(\boldsymbol{D}^k - \boldsymbol{D}^{k+1})K^{(l_2,l_2)}(\boldsymbol{L}^{(\alpha)})\sigma^{l_1,\alpha}(\boldsymbol{L}^{(\alpha)})\pi^{k,l_1}(\boldsymbol{L}^{(\alpha)})$$
$$-\frac{l_2!}{(\alpha+1)_{l_2}}\boldsymbol{A}_0^{(\alpha,l_1)}\left(\sum_{k=1}^{l_2+1}\boldsymbol{D}^k\sigma^{l_2,\alpha}(\boldsymbol{L}^{(\alpha)})\pi^{k,l_2}(\boldsymbol{L}^{(\alpha)})\right)$$
$$+\frac{l_2!}{(\alpha+1)_{l_2}}\sum_{k=1}^{l_2+1}\boldsymbol{D}^k\alpha^{(\alpha,l_1)}(\boldsymbol{L}^{(\alpha)})\sigma^{l_2,\alpha}(\boldsymbol{L}^{(\alpha)})\pi^{k,l_2}(\boldsymbol{L}^{(\alpha)}).$$

If $\alpha \in \mathbb{N}$, then the operator $\boldsymbol{V}^{(\alpha,l_1,l_2)}$ is at most of the order $4\alpha + 4l_1 + 4l_2 + 5$. Similarly, for the order of the operator $\boldsymbol{V}^{(\alpha,l_2,l_1)}$ mapping $L_n^{(\alpha)}(x)$ into $-(\boldsymbol{A}_0^{(\alpha,l_2)} - \alpha_n^{(\alpha,l_2)}\boldsymbol{I})Q_n^{\alpha}(x,l_1)$ the same statement is true. The order of $\gamma^{(\alpha,l_1,l_2)}(\boldsymbol{L}^{(\alpha)})$ is infinite if $\alpha \notin \mathbb{N}$.

If $\alpha \in \mathbb{N}$, then the order of $\gamma^{(\alpha,l_1,l_2)}(\boldsymbol{L}^{(\alpha)})$ is $4\alpha + 4l_1 + 4l_2 + 6$, implying that

$$\boldsymbol{C}_0^{(\alpha,l_1,l_2)} = \gamma^{(\alpha,l_1,l_2)}(\boldsymbol{L}^{(\alpha)}) + \boldsymbol{U}^{(\alpha,l_1,l_2)} + \boldsymbol{V}^{(\alpha,l_1,l_2)} + \boldsymbol{V}^{(\alpha,l_2,l_1)} \tag{22}$$

is of order $4\alpha + 4l_1 + 4l_2 + 6$ in this case. Further by (15)

$$c_{4\alpha+4l_1+4l_2+6,0}(x;\alpha,l_1,l_2)$$
$$= \frac{-(\alpha!)^2(l_2-l_1)^2x^{2\alpha+2l_1+2l_2+3}}{(\alpha+2l_1+1)(\alpha+2l_2+1)(2\alpha+2l_1+2l_2+3)[(\alpha+l_1+l_2+1)(\alpha+l_1)!(\alpha+l_2)!]^2}.$$

6. An interesting property of the operators

If $\alpha \in \mathbb{N}$, then the operators $\boldsymbol{A}_0^{(\alpha,l)}$ with $l \in \{l_1, l_2\}$ and $\boldsymbol{C}_0^{(\alpha,l_1,l_2)}$ have the remarkable property that the sum of their coefficients is equal to zero. In fact, we will show that

$$\sum_{i=1}^{\infty} a_{i,0}(x;\alpha,l) = 0, \quad \sum_{i=1}^{\infty} c_{i,0}(x;\alpha,l_1,l_2) = 0.$$

In the case, $l_1 = 0$, $l_2 = 1$ this has been pointed out by Koekoek [10] for $\alpha \in \{0,1,2\}$. For the coefficients $\{a_{i,0}(x;\alpha,l)\}_{i=1}^{\infty}$ this was proved in [9] in the cases $l=0$ and $l=1$, and the general case was treated in [4]. For the coefficients $\{c_{i,0}(x;\alpha,l_1,l_2)\}_{i=1}^{\infty}$ this was shown in [9] in the case $l_1=0$, $l_2=1$. Here an easy new proof in the general case is given. It is necessary and sufficient to show that $\boldsymbol{A}_0^{(\alpha,l)}\mathrm{e}^x=0$ and $\boldsymbol{C}_0^{(\alpha,l_1,l_2)}\mathrm{e}^x=0$. In [10] it was already pointed out that $\boldsymbol{L}^{(\alpha)}\mathrm{e}^x=-(\alpha+1)\mathrm{e}^x$, which implies that for any polynomial $p(x)$ we have

$$p(\boldsymbol{L}^{(\alpha)})\mathrm{e}^x = p(-\alpha-1)\mathrm{e}^x.$$

If we consider (21), $\boldsymbol{A}_0^{(\alpha,l)}\mathrm{e}^x = 0$ follows immediately, since $\alpha^{(\alpha,l)}(\boldsymbol{L}^{(\alpha)})\mathrm{e}^x = \alpha^{(\alpha,l)}(-\alpha-1)\mathrm{e}^x = 0$ and the second term gives 0, due to the factor $\boldsymbol{D}^k - \boldsymbol{D}^{k+1}$. Now we consider (22): $\gamma^{(\alpha,l_1,l_2)}(\boldsymbol{L}^{(\alpha)})\mathrm{e}^x = \gamma^{(\alpha,l_1,l_2)}(-\alpha-1)\mathrm{e}^x = 0$, further $\boldsymbol{U}^{(\alpha,l_1,l_2)}\mathrm{e}^x = 0$, due to the factor $\boldsymbol{D}^k - \boldsymbol{D}^{k+1}$ and for the operators $\boldsymbol{V}^{(\alpha,l_1,l_2)}$ and $\boldsymbol{V}^{(\alpha,l_2,l_1)}$, the first term working on e^x yields 0, due the factor $\boldsymbol{D}^k - \boldsymbol{D}^{k+1}$, the second term gives 0, since $\boldsymbol{A}_0^{(\alpha,l)}\mathrm{e}^x=0$ and the third gives 0 because $\alpha^{(\alpha,l)}(\boldsymbol{L}^{(\alpha)})\mathrm{e}^x=0$. Hence $\boldsymbol{C}_0^{(\alpha,l_1,l_2)}\mathrm{e}^x=0$.

7. Other choices of the arbitrary values

In [5, Section 4], the general form of the operators $\boldsymbol{A}^{(\alpha,l_1)}, \boldsymbol{A}^{(\alpha,l_2)}, \boldsymbol{C}^{(\alpha,l_1,l_2)}$ is given. If for $m \in \{1,2,3,\ldots\}$ we define the operators $\boldsymbol{J}^{(\alpha,m)}$ and $\boldsymbol{K}^{(\alpha,m)}$ by

$$\boldsymbol{J}^{(\alpha,m)}L_n^{(\alpha)}(x) = \delta_{n,m}L_n^{(\alpha)}(x) \quad \text{for all } n \in \mathbb{N}$$

and

$$\boldsymbol{K}^{(\alpha,m)}L_n^{(\alpha)}(x) = \begin{cases} 0 & \text{for all } n \in \{0,1,\ldots,m-1\}, \\ L_n^{(\alpha)}(x) & \text{for all } n \in \{m,m+1,m+2,\ldots\} \end{cases}$$

then the operators $\boldsymbol{A}^{(\alpha,l_1)}$ and $\boldsymbol{A}^{(\alpha,l_2)}$ can be put in the form

$$\boldsymbol{A}^{(\alpha,l_1)} = \boldsymbol{A}_0^{(\alpha,l_1)} + \sum_{m=1}^{l_1-1} \alpha_m^{(\alpha,l_1)}\boldsymbol{J}^{(\alpha,m)} + \alpha_{l_1}^{(\alpha,l_1)}\boldsymbol{K}^{(\alpha,l_1)},$$

$$\boldsymbol{A}^{(\alpha,l_2)} = \boldsymbol{A}_0^{(\alpha,l_2)} + \sum_{m=1}^{l_2-1} \alpha_m^{(\alpha,l_2)}\boldsymbol{J}^{(\alpha,m)} + \alpha_{l_2}^{(\alpha,l_2)}\boldsymbol{K}^{(\alpha,l_2)}.$$

Here $\{\alpha_n^{(\alpha,l_1)}\}_{n=1}^{l_1}$ (if $l_1 > 0$) and $\{\alpha_n^{(\alpha,l_2)}\}_{n=1}^{l_2}$ are the chosen values. The general form of the operator $\boldsymbol{C}^{(\alpha,l_1,l_2)}$ is more complicated, but also contains the operators $\boldsymbol{J}^{(\alpha,m)}$ and $\boldsymbol{K}^{(\alpha,m)}$. In [2], it has already been pointed out that for all $m \in \{1,2,3,\ldots\}$ the operators $\boldsymbol{J}^{(\alpha,m)}$ and $\boldsymbol{K}^{(\alpha,m)}$ are both of infinite order, which implies that any other choice than 0 for one of the arbitrary values will lead to an operator of infinite order.

8. Conjecture

Let $k \in \{1,2,3,\ldots\}$ and let $l_1 < l_2 < \cdots < l_k$ be nonnegative integers. Let for $s \in \{1,2,\ldots,k\}$ an arbitrary subset τ of $\{1,2,\ldots,k\}$ with s elements be given by $\tau = \{j_1, j_2, \ldots, j_s\}$, where $j_1 < j_2 < \cdots < j_s$. Consider the determinant of order $s+1$

$$\Delta_n(x;\tau) = \begin{vmatrix} L_n^{(\alpha)}(x) & K_{n-1}^{(0,l_{j_1})}(x,0) & K_{n-1}^{(0,l_{j_2})}(x,0) & \ldots & K_{n-1}^{(0,l_{j_s})}(x,0) \\ (-1)^{l_{j_1}}\binom{n+\alpha}{n-l_{j_1}} & K_{n-1}^{(l_{j_1},l_{j_1})}(0,0) & K_{n-1}^{(l_{j_1},l_{j_2})}(0,0) & \ldots & K_{n-1}^{(l_{j_1},l_{j_s})}(0,0) \\ (-1)^{l_{j_2}}\binom{n+\alpha}{n-l_{j_2}} & K_{n-1}^{(l_{j_2},l_{j_1})}(0,0) & K_{n-1}^{(l_{j_2},l_{j_2})}(0,0) & \ldots & K_{n-1}^{(l_{j_2},l_{j_s})}(0,0) \\ \ldots & \ldots & \ldots & \ldots & \ldots \\ (-1)^{l_{j_s}}\binom{n+\alpha}{n-l_{j_s}} & K_{n-1}^{(l_{j_s},l_{j_1})}(0,0) & K_{n-1}^{(l_{j_s},l_{j_2})}(0,0) & \ldots & K_{n-1}^{(l_{j_s},l_{j_s})}(0,0) \end{vmatrix}.$$

Then we conjecture that the polynomials $\{L_n^{\alpha,M_1,M_2,\ldots,M_k}(x,l_1,l_2,\ldots,l_k)\}_{n=0}^{\infty}$, given by

$$L_n^{\alpha,M_1,M_2,\ldots,M_k}(x,l_1,l_2,\ldots,l_k) = L_n^{(\alpha)}(x) + \sum_{\tau}\left[\left(\prod_{i=1}^{s} M_{j_i}\right)\Delta_n(x;\tau)\right]$$

are orthogonal with respect to

$$\Phi_k(p,q) = \frac{1}{\Gamma(\alpha+1)}\int_0^\infty p(x)q(x)x^\alpha \mathrm{e}^{-x}\,\mathrm{d}x + \sum_{i=1}^{k} M_i p^{(l_i)}(0)q^{(l_i)}(0).$$

Further, we conjecture that the polynomials $\{\Delta_n(x;\tau)\}_{n=j_s}^{\infty}$ are eigenfunctions of a linear differential operator $\boldsymbol{A}_\tau$, which in the case $\alpha \in \mathbb{N}$ and all the arbitrary eigenvalues are chosen to be 0, is of order $2s(\alpha+1)+2+4\sum_{i=1}^{s} l_{j_i}$. The polynomials $\{L_n^{\alpha,M_1,M_2,\ldots,M_k}(x,l_1,l_2,\ldots,l_k)\}_{n=0}^{\infty}$ are eigenfunctions of a linear differential operator

$$\boldsymbol{L}^{(\alpha)} + \sum_{\tau}\left[\left(\prod_{i=1}^{s} M_{j_i}\right)\boldsymbol{A}_\tau\right]$$

which, if $M_i > 0$ for all $i \in \{1,2,\ldots,k\}$, is of order $2k(\alpha+1)+2+4\sum_{i=1}^{k} l_i$.

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